5.2 Homework Solutions
Math 206
Mammenga

THE AREA FUNCTION

ASSIGNMENT:

1. Find the area functions of each of the following functions, letting $a = 2$.
   (a) $f(t) = -2$
   Solution: By graphing the function, we see that by choosing a value of t to the right of $a = 2$ yields an area of $(2)(t-2)$, and since this area is below the x-axis and since we are moving left-to-right, it should have a negative sign. Therefore, $A_f(t) = -2(t-2)$. Note that if you had selected a value of t that was to the left of $a = 2$, then the area is $(2)(2-t)$, and since the area is below the x-axis and since we are moving right-to-left, it should be a positive value. Hence, $A_f(t) = 2(2-t)$ (which is exactly the same as what we found before, meaning they are the same function).
   (b) $g(t) = \frac{3}{4}t - 5$
   Solution: By choosing a value of t to the right of $a = 2$ but to the left of the root of the graph, we see that the area is comprised of a rectangle and a triangle. The rectangle has area $(t-2)(3/4t-5)(-1)$ (I am multiplying by -1 since I want the *lengths* of each side to be positive and the expression $3/4t-5$ is negative). Since this area is below the x-axis and since we are moving left-to-right, it should be negative. So, the rectangle contributes a value of $-(t-2)(3/4t-5)(-1) = (t-2)(3/4t-5)$. Now, the area of the triangle is $1/2(t-2)((3/4t-2)-(-7/2))$. Then lengths of both the base and the height of this triangle are positive, so we do not need to multiply by -1. Since the area is below the x-axis and we are moving left-to-right, the triangle contributes $-1/2(t-2)(3/4t+3/2)$. Therefore,

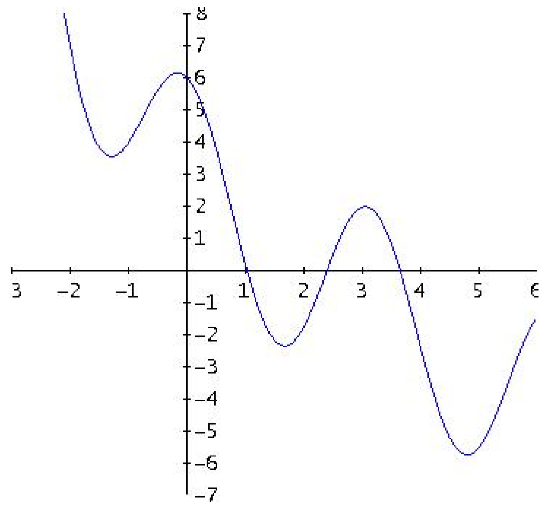
$$A_g(t) = (t-2)(3/4t-5) - 1/2(t-2)(3/4t+3/2) = \frac{3}{8}t^2 - 5t + \frac{17}{2}.$$

   (c) $h(t) = -2t + 1$
   Solution: By choosing a value of t to the right of $a = 2$, we see that the area is comprised of a rectangle and a triangle. The rectangle has area $(t-2)(3)$. This area is below the x-axis and we are moving left-to-right, so it has a negative sign, meaning $-3(t-2)$. The triangle has area $1/2(t-2)((-3)-(-2t+1))$, and again this should be negative, so $-1/2(t-2)(2t-4)$. Therefore,

$$A_h(t) = -3(t-2) - 1/2(t-2)(2t-4) = -t^2 + t - 2.$$

2. Plot $f(t) = e^{-4/5t} + 3\cos(2t) - t + 2$ using Derive, and let

$$A(x) = \int_0^x f(t)dt \qquad \text{and} \qquad B(x) = \int_2^x f(t)dt.$$



(a) Which is greater, $A(2)$ or $A(5)$?
Solution: Recall that $A(x)$ is the area function that starts at 0 and ends at x. Since we are comparing $A(2)$ and $A(5)$, we only have to determine whether the area between 2 and 5 totals to a positive value or a negative value. This is because $A(2)$ and $A(5)$ share the area from 0 to 2. The graph shows that there is more area below the x-axis than above the x-axis over the interval $[2, 5]$, meaning $A(5)$ will have added more *negative* area than $A(2)$. Therefore, $A(2)$ is greater.

(b) Which is greater, $B(1)$ or $B(3)$?
Solution: Recall that $B(x)$ is the area function that starts at 2 and ends at x. For $B(1)$, we are moving right-to-left, and the area is completely under the x-axis. Both of these facts contribute a negative sign to the area, meaning the end result is a positive area. For $B(3)$, we are starting at 2 and ending at 3, so we are moving left-to-right. Since part of the area is below the x-axis and part above, the difference between them is the value of $B(3)$. Hence, $B(1)$ is greater than $B(3)$.

(c) Find a value c such that $B(c) = 0$.
Solution: As noted in the textbook and our lecture notes, the area function will have a value of 0 if we evaluate it at the same value as a (the starting position of the area function). Since $B(x)$ starts at $a = 2$, we know that $B(2) = 0$. Hence, $c = 2$.

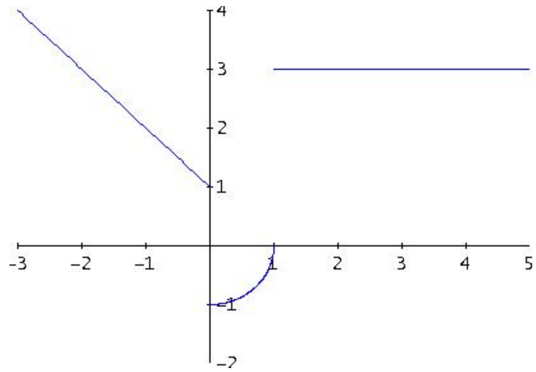
(d) Is $A(x)$ concave up or concave down over the interval $[0, 1]$?
Solution: Since the function is decreasing over $[0, 1]$, $A(x)$ is concave down.

(e) Is $B(x)$ increasing or decreasing over the interval $[2, 3]$?
Solution: This is a poorly worded question. Since the function is negative over the interval $[2, \approx 2.4]$ and positive over the interval $[\approx 2.4, 3]$, it is both increasing and decreasing.

(f) List the following values in *increasing* order: $A(1)$, $A(2)$, $B(-2)$, and $B(1)$.
Solution: $B(-2)$, $A(2)$, $B(1)$, $A(1)$ (Recall that $B(1)$ and $B(-2)$ move right-to-left.) I would also accept $B(-2)$, $B(1)$, $A(2)$, $A(1)$ since it's not totally obvious which is greater between $B(1)$ and $A(2)$.

3. Let $f(t)$ be the following piece-wise defined function:

$$f(t) = \begin{cases} -t+1, & \text{if } t \leq 0; \\ -\sqrt{1-t^2}, & \text{if } 0 < t < 1; \\ 3, & \text{if } t \geq 1. \end{cases}$$



(a) Compute $\displaystyle\int_0^{-2} f(t)dt$.

Solution: This region is comprised of a rectangle and a triangle. The rectangle has area (1)(2) and the triangle has area $1/2(3-1)(2) = 2$. Since this integral moves right-to-left and since the region is above the x-axis, the value should have a negative sign. Hence,

$$\int_0^{-2} f(t)dt = -(2+2) = -4.$$

(b) Compute $\displaystyle\int_{-1}^{1} f(t)dt$.

Solution: This region is comprised of a rectangle, a triangle, and a quarter-circle. The rectangle has area $(1)(1) = 1$, the triangle has area $1/2(1)(1) = 1/2$, and the quarter-circle has area $1/4 \cdot \pi(1)^2 = \frac{\pi}{4}$. Now, since the quarter-circle is below the x-axis and the other two subregions are above the x-axis, we have

$$\int_{-1}^{1} f(t)dt = 1 + \frac{1}{2} - \frac{\pi}{4} = \frac{3}{2} - \frac{\pi}{4}.$$

(c) Compute $\displaystyle\int_0^4 f(t)dt$.

Solution: This region is comprised of a quarter-circle and a rectangle. We have already established that the area of the quarter-circle is $\frac{\pi}{4}$. The area of the rectangle is $(3)(3) = 9$. Taking into account the sign of each subregion, we have

$$\int_0^4 f(t)dt = 9 - \frac{\pi}{4}.$$