4.6 Homework Solutions
Math 206
Mammenga

NEWTON'S METHOD: FINDING ROOTS

ASSIGNMENT:

1. Consider the function $f(x) = (x+3)(x-2)$. Using $x_0 = 3$, find the first three Newton estimates of the root $x = 2$, and state the number of decimal places of accuracy for each.

   Solution: Using the formula for Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we have

$$\begin{aligned}
x_0 &= 3 \\
x_1 &= 3 - \frac{f(3)}{f'(3)} = 3 - \frac{6}{7} = 2.142857142 \\
x_2 &= 2.142857142 - \frac{f(2.142857142)}{f'(2.142857142)} = 2.142857142 - \frac{0.7346938775}{5.285714285} = 2.003861003 \\
x_3 &= 2.003861003 - \frac{f(2.003861003)}{f'(2.003861003)} = 2.003861003 - \frac{0.0193199223}{5.007722005} = 2.000002976.
\end{aligned}$$

   Since we know that the exact value of the root is 2, we see that x_1 only has the digit "2" accurate, x_2 has 2 decimal places of accuracy, and x_3 has 5 decimal places of accuracy.

2. Let $g(x) = \frac{x}{3}(\sin(x^2))(\tan(x))$. Estimate the root that is between 1 and 2, accurate to 6 decimal places. You must show your work for full credit.

   Solution: A picture of the graph of $g(x)$ can viewed on the corresponding jpg. My initial guess for the root between 1 and 2 is 1.7, so I will let $x_0 = 1.7$. Then,

$$\begin{aligned}
x_0 &= 1.7 \\
x_1 &= 1.7 - \frac{f(1.7)}{f'(1.7)} = 1.748861876 \\
x_2 &= 1.748861876 - \frac{f(1.748861876)}{f'(1.748861876)} = 1.770061338 \\
x_3 &= 1.770061338 - \frac{f(1.770061338)}{f'(1.770061338)} = 1.772429507 \\
x_4 &= 1.772429507 - \frac{f(1.772429507)}{f'(1.772429507)} = 1.772453848 \\
x_5 &= 1.772453848 - \frac{f(1.772453848)}{f'(1.772453848)} = 1.772453851.
\end{aligned}$$

3. Apply Newton's method to $f(x) = e^{-x}$, letting $x_0 = 0$.

   (a) Find the first five Newton estimates.

Solution: Again, using the formula, we have

$$\begin{aligned}
x_0 &= 0 \\
x_1 &= 0 - \frac{f(0)}{f'(0)} = -\frac{1}{-1} = 1 \\
x_2 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e^{-1}}{-e^{-1}} = 2 \\
x_3 &= 2 - \frac{f(2)}{f'(2)} = 2 - \frac{e^{-2}}{-e^{-2}} = 3 \\
x_4 &= 3 - \frac{f(3)}{f'(3)} = 3 - \frac{e^{-3}}{-e^{-3}} = 4 \\
x_5 &= 4 - \frac{f(4)}{f'(4)} = 4 - \frac{e^{-4}}{-e^{-4}} = 5.
\end{aligned}$$

(b) How many iterations (repetitions) are necessary to have an estimate that is accurate to 2 decimal places?

Solution: Recall that Newton's method is a process to come up with an estimate for the root of a function. Where the actual root of the function $f(x) = e^{-x}$? This function has no roots since it never crosses the x-axis! Therefore, we are trying to estimate a value that doesn't exist and we will *never* get an estimate that is accurate to 2 decimal places.