4.5 Homework Solutions
Math 206
Mammenga

RELATED RATES

ASSIGNMENT:

1. The base of a right triangle is increasing at a rate of 3 cm/sec and its height is decreasing at a rate of 2 cm/sec. At what rate is the area increasing/decreasing when the base has a length of 10 cm and the height has a length of 14 cm?

   Solution: We are concerned with the rate of change of the area of the triangle, so we must differentiate the equation for area of a triangle:

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + b\frac{dh}{dt}\right)$$

   We are considering the moment in time when $b = 10$ and $h = 14$. We also know that $\frac{db}{dt} = 3$ and $\frac{dh}{dt} = -2$. Putting this information into the previous equation, we have

$$\frac{dA}{dt} = \frac{1}{2}\left((3)(14) + (10)(-2)\right) = 11.$$

   This means that the area is *increasing* at a rate of 11 cm^2 per second.

2. A highway patrol plane flies 1 mile above a straight section of rural interstate highway at a steady ground speed of 150 miles per hour. The pilot sees an oncoming car and determines that the line-of-sight distance from the plane to the car is 1.2 miles and that this distance is decreasing at a rate of 115 miles per hour. What is the speed of the car?

   Solution: This situation can be modeled by a right triangle, where the hypotenuse is the distance between the plane in the sky and the car on the highway. Let a be the side that is between the plane and the ground (and is perpendicular to the ground), let b be the side that is along the highway with the car at one end, and let c be the hypotenuse. Then

$$a^2 + b^2 = c^2,$$

   which implies that

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}.$$

   Now, what information do we know that we can substitute into this equation? We are given that a is 1 mile and that c is 1.2 miles. Also, we know that $\frac{dc}{dt} = 115$ miles per hour. By the Pythagorean theorem, we can find the length of b, which turns out to be $\sqrt{.44}$ miles. Also, the height of the plane is not changing, so we know that $\frac{da}{dt} = 0$. Putting this together, we have

$$2(1)(0) + 2(\sqrt{.44})\frac{db}{dt} = 2(1.2)(-115),$$

which implies

$$\frac{db}{dt} = -208.04284.$$

Now, the change in the length of b is from *both* the speed of the plane and the speed of the car. Since we are given that the plane is traveling at a ground speed of 150 miles per hour, we can conclude that the car is traveling at $|-208-(-150)| = 58$ miles per hour. (I used absolute values since we are finding a speed, which is ALWAYS positive, as opposed to velocity.)

3. You are filling a hemispherical pool with water from a hose at a constant rate. You notice that the depth of the water is increasing at 1 in/min when the depth of the water is 18 inches. If the radius of the pool is 5 feet, how fast is the water entering the pool from the hose (in cubic inches per second)? You will need to use the fact that the volume of the bottom portion of a sphere with radius r is given by

$$V = \pi D^2 \left(r - \frac{1}{3}D\right),$$

where D is the depth of the bottom portion of the sphere.

Solution: We are supposed to find the rate at which the volume is increasing (i.e. $\frac{dV}{dt}$). That means we need to differentiate the volume formula:

$$\frac{dV}{dt} = \pi\left(2D\frac{dD}{dt}\left(r - \frac{1}{3}D\right) + D^2\left(\frac{dr}{dt} - \frac{1}{3}\frac{dD}{dt}\right)\right).$$

We just need to substitute in the values that we know. We are given that $D = 18$ and that $r = 60$ (after converting to inches). Also, $\frac{dD}{dt} = 1$ in/min, but since we are supposed to give our answer in terms of seconds, we can write $\frac{dD}{dt} = \frac{1}{60}$ in/sec. What is $\frac{dr}{dt}$? Recall that r stands for the length of the radius of the *original* sphere. So, no matter what level the water has risen to, the r-value is still 60 inches. Therefore, $\frac{dr}{dt} = 0$. Hence,

$$\frac{dV}{dt} = \pi\left(2(18)\frac{1}{60}\left(60 - \frac{1}{3}(18)\right) + (18)^2\left(0 - \frac{1}{3}\cdot\frac{1}{60}\right)\right) = 96.13273519.$$

Therefore, the water is entering the pool from the hose at a rate of 96.13 cubic inches per second.