3.4 Homework Solutions
Math 206
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INVERSE FUNCTIONS AND THEIR DERIVATIVES; INVERSE TRIGONOMETRIC FUNCTIONS

ASSIGNMENT:

1. Compute the derivative of each of the following functions.
   (a) $f(x) = \arcsin\left(\frac{x}{2}\right)$
   Solution: Using the formula above for the derivative of the inverse sine function and the formula for the chain rule, we get
   $$f'(x) = \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{1-x^2}}.$$
   (b) $f(x) = \ln(7x^2 - \text{arcsec}x)$
   Solution: Again, we will need to use the chain rule and the formula for the derivative of the inverse secant function:
   $$f'(x) = \frac{1}{7x^2 - \text{arcsec}x}\left(14x - \frac{1}{|x|\sqrt{x^2-1}}\right).$$
   (c) $f(x) = \dfrac{x^3+2}{\sin(\arctan x)}$
   Solution: This time, we use the quotient rule and the formula for the derivative of the inverse tangent function:
   $$f'(x) = \frac{(3x^2)(\sin(\arctan x)) - (x^3+2)(\cos(\arctan x))\left(\frac{1}{1+x^2}\right)}{\sin^2(\arctan x)}.$$
   Anytime that we have an inverse trig function composed with another trig function, we should sketch a triangle to find the value. If $\arctan x = \theta$, where θ is some angle in a right triangle that is not the right-angle, then that means that $\tan\theta = x$. We can then call x the side opposite of θ and we can call 1 the side that is adjacent to θ. The third side must have length $\sqrt{1+x^2}$. Now, $\sin(\arctan x) = \sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\arctan x) = \cos\theta = \frac{1}{\sqrt{1+x^2}}$. Hence,
   $$f'(x) = \frac{(3x^2)\frac{x}{\sqrt{1+x^2}} - (x^3+2)\frac{1}{\sqrt{1+x^2}}\left(\frac{1}{1+x^2}\right)}{\left(\frac{x}{\sqrt{1+x^2}}\right)^2},$$
   which is equivalent to
   $$f'(x) = \frac{\frac{3x^3}{\sqrt{1+x^2}} - \frac{x^3+2}{(1+x^2)^{3/2}}}{\frac{x^2}{1+x^2}}.$$
   You could simplify this more if you wanted.

2. Where is the graph of $\arctan x$ increasing? Where is the graph of $\arctan x$ concave up? You must explain how you got your answers, and it should involve taking a derivative (or two).

Solution: A graph is increasing wherever its corresponding derivative is positive, and a graph is concave up wherever its corresponding second derivative is positive. So, since $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$ is positive for all values of x, we can conclude that $\arctan x$ is increasing for all values of x. The second derivative of $\arctan x$ is given by

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2}.$$

Since the denominator of this fraction is positive for all values of x, we only need to determine when the numerator is positive. Hence, $-2x > 0$ implies $x < 0$ (remember to switch the direction of an inequality symbol whenever multiplying or dividing by a negative!). Therefore, $\arctan x$ is concave up for all negative values of x.

3. Suppose that $f(x)$ is a differentiable function and that its inverse, $f^{-1}(x)$ exists. Is the derivative of the inverse function equal to the inverse of the derivative function? That is, do we have the equality

$$\left(f^{-1}\right)'(x) = \left(f'\right)^{-1}(x)?$$

If so, explain why. If not, give an example of a function where the two are not equal.

Solution: We will demonstrate that this equality does not hold (is not true) by demonstrating an example that doesn't work. Let $f(x) = (x+1)^3$. Then $f'(x) = 3(x+1)^2$ and $f^{-1}(x) = -1 + \sqrt[3]{x}$. We will show that the inverse of $f'(x) = 3(x+1)^2$ is not equal to the derivative of $f^{-1}(x) = -1 + \sqrt[3]{x}$. First,

$$\left(f'\right)^{-1}(x) = -1 + \pm\sqrt{\frac{x}{3}}.$$

Second,

$$\left(f^{-1}\right)'(x) = \frac{1}{3}x^{-2/3}.$$

Clearly these are not the same functions!