3.1 Homework Solutions
Math 206
Mammenga

THE PRODUCT AND QUOTIENT RULES

ASSIGNMENT:

1. Find $f'(2)$ if $f(x) = 3xe^x$. You must show your work for full credit.

   Solution: We must use the product rule, where $3x$ is the first function and e^x is the second function. And so,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x) \cdot e^x + (3x) \cdot \frac{d}{dx}\left(e^x\right) \\ &= 3e^x + 3xe^x \\ &= 3e^x(1+x). \end{aligned}$$

   Now, evaluating the derivative at 2 yields

$$f'(2) = 3e^2(1+2) = 9e^2.$$

2. Let $g(x) = \frac{x^2 \sin x}{\ln x}$. Find $g'(x)$. You must show your work for full credit.

   Solution: To find the derivative, we need to use the quotient rule:

$$\begin{aligned} g'(x) &= \frac{\frac{d}{dx}\left(x^2 \sin x\right) \cdot (\ln x) - \left(x^2 \sin x\right) \cdot \frac{d}{dx}(\ln x)}{(\ln x)^2} \\ &= \frac{\left(2x \cdot \sin x + x^2 \cdot \cos x\right)(\ln x) - \left(x^2 \sin x\right)\frac{1}{x}}{(\ln x)^2} \\ &= \frac{2x \sin x \ln x + x^2 \cos x \ln x - x \sin x}{(\ln x)^2}. \end{aligned}$$

3. Find (exactly) all stationary points and inflection points on the interval $(1, \infty)$ for the function $f(x) = \dfrac{\ln x}{x}$.

   Solution: A stationary point on the graph of a function occurs where the first derivative of the function equals zero. So, let's first compute $f'(x)$. This involves the quotient rule.

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(\ln x) \cdot x - (\ln x) \cdot \frac{d}{dx}(x)}{x^2} \\ &= \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} \\ &= \frac{1 - \ln x}{x^2}. \end{aligned}$$

   In order for this fraction (the first derivative) to equal zero, we just need the numerator to equal zero. That means we have the equation $0 = 1 - \ln x$, which is equivalent to $\ln x = 1$. Hence, $x = e$. To find the y-coordinate that corresponds to this x-coordinate, we just need to plug in $x = e$ into $f(x)$.

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

We may now conclude that the only stationary point on the graph of $f(x)$ occurs at $\left(e, \frac{1}{e}\right)$.

An inflection point on the graph of a function occurs where the second derivative of the function equals zero. So, we need to find $f''(x)$:

$$\begin{aligned} f''(x) &= \frac{\frac{d}{dx}(1-\ln x)\cdot x^2 - (1-\ln x)\cdot \frac{d}{dx}(x^2)}{(x^2)^2} \\ &= \frac{\frac{-1}{x}\cdot x^2 - (1-\ln x)\cdot 2x}{x^4} \\ &= \frac{-x-2x+2x\ln x}{x^4} \\ &= \frac{-3+2\ln x}{x^3}. \end{aligned}$$

In order for this fraction (the second derivative) to equal zero, we just need the numerator to equal zero. That means we have the equation $0 = -3 + 2\ln x$, which is equivalent to $\ln x = \frac{3}{2}$. Hence, $x = e^{3/2}$. To find the y-coordinate that corresponds to this x-coordinate, we just need to plug in $x = e^{3/2}$ into $f(x)$.

$$f\left(e^{3/2}\right) = \frac{\ln e^{3/2}}{e^{3/2}} = \frac{3}{2e^{3/2}}$$

We may now conclude that the only inflection point on the graph of $f(x)$ occurs at $\left(e^{3/2}, \frac{3}{2e^{3/2}}\right)$.

4. At what values of x will the tangent lines to the function $g(x) = \dfrac{x+2}{x-2}$ be parallel to the line $x + 2y = 1$?

Solution: Recall that two lines are parallel if and only if their slopes are equal. If we re-arrange $x+2y=1$, we get $y = -\frac{1}{2}x + \frac{1}{2}$. This means that this line has a slope of $-\frac{1}{2}$. In order for the tangent line to $g(x)$ to have a slope of $-\frac{1}{2}$, we need $g'(x)$ to equal $-\frac{1}{2}$. So, let's compute $g'(x)$:

$$\begin{aligned} g'(x) &= \frac{\frac{d}{dx}(x+2)\cdot(x-2) - (x+2)\cdot\frac{d}{dx}(x-2)}{(x-2)^2} \\ &= \frac{1\cdot(x-2)-(x+2)\cdot 1}{(x-2)^2} \\ &= \frac{-4}{(x-2)^2}. \end{aligned}$$

Setting this derivative equal to $-\frac{1}{2}$ yields

$$\begin{aligned} -\frac{1}{2} &= \frac{-4}{(x-2)^2} \\ 8 &= (x-2)^2 \\ \pm 2\sqrt{2} &= x-2 \\ 2 \pm 2\sqrt{2} &= x. \end{aligned}$$

Therefore, at the x-values of $2 \pm 2\sqrt{2}$, the tangent line of $g(x)$ will be parallel to the line $x + 2y = 1$.